Theory of degenerate three-wave mixing using circuit QED in solid-state circuits

Ye Cao (曹业),[1] Wen Yi Huo (霍文义),[2] Qing Ai (艾清),[3] and Gui Lu Long (龙桂鲁)[1,4]

[1]*Key Laboratory of Atomic and Molecular Nanosciences and Department of Physics, Tsinghua University, Beijing 100084, China*

[2]*Institute of Applied Physics and Computational Mathematics, Beijing 100088, China*

[3]*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

[4]*Tsinghua National Laboratory For Information Science and Technology, Beijing 100084, China*

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We study the theory of degenerate three-wave mixing and the generation of squeezed microwaves using circuit quantum electrodynamics in solid state circuits. The Hamiltonian for degenerate three-wave mixing, which seemed to be given phenomenologically in quantum optics, is derived by quantum mechanical calculations. The nonlinear medium needed in three-wave mixing is composed of a series of superconducting charge qubits which are located inside two superconducting transmission-line resonators. Here, the multiqubit ensemble is present to enhance the effective coupling constant between the two modes in the transmission-line resonators. In the squeezing process, the qubits are kept in their ground states so that their decoherence does not corrupt the squeezing. The main obstacle preventing a large squeezing efficiency is the decay rate of the transmission-line resonator.

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I. INTRODUCTION

Squeezed states provide a good example of the interplay between experiment and theory in the development of quantum mechanics [1]. The possibility of using squeezed states in quantum communication and of applying squeezed states to the study of fundamental quantum phenomena as well as to detecting gravitational radiation has been recognized [2–6]. Degenerate three-wave mixing and four-wave mixing are the two main means used to generate squeezed light in quantum optics [7,8]. In condensed matter physics, the theoretical study and experimental demonstration of the generation of squeezed microwaves via degenerate four-wave mixing have been successfully accomplished by using a Josephson junction parametric amplifier [9–11].

Recently, there has been significant progress in simulating quantum optics phenomena in superconducting solid state electrical circuits; the so-called circuit quantum electrodynamics (circuit QED). The strong couplings between a one-dimensional superconducting transmission-line resonator (TLR) and a superconducting charge qubit, an LC oscillator, and a superconducting flux qubit have been experimentally realized [12,13]. Many phenomena in quantum optics and quantum mechanics, such as dressed states, sideband transitions, single-atom lasing, the Berry phase, and the photon bunching effect, have been observed in circuit QED [14–19]. The experiment of resolving photon number states indicates that the coupling in circuit-QED systems can achieve the extremely-strong-coupling limit [20]. Recent progress in experiments [21–23] showed the potential to extend the coupling between the boson mode and few qubits to multiqubits. On the other hand, in our previous paper [24], we considered the possibility of creating quantum entanglement of two electron spins by coupling them simultaneously to a many-spin ensemble. It was discovered that the effective coupling constant was increased by a factor of $\sqrt{N}$ when an electron spin was coupled with an ensemble of N nuclear spins. Similar enhanced coupling is also expected and has already been confirmed experimentally [21] in the case of the interaction between a boson mode and a many-spin ensemble.

Based on these experimental advances [25], it is possible to obtain strong nonlinear interactions in circuit-QED systems. A phase-preserving amplifier based on the nonlinear interaction was proposed and realized to improve its performance in the regime near the quantum limit [26]. More recently, there have been promising efforts to realize squeezed states of nanomechanical resonators [27–30] and microwaves using circuit QED [31,32]. As is well known, a nonlinear medium is indispensable to generate squeezed states. In quantum optics, theory of and experiments in degenerate three-wave mixing are well developed for generating squeezed states [33–35]. However, in text books for quantum optics [1,36,37], the degenerate three-wave mixing Hamiltonian seems to be given phenomenologically. Motivated by the strong nonlinearity, we study in this paper the theory of degenerate three-wave mixing to generate squeezed microwaves using circuit QED. The system considered consists of two one-dimensional superconducting TLRs and a series of superconducting charge qubits to enhance the effective coupling constant between two bosonic modes. The nonlinear medium needed in three-wave mixing can be constructed by the series of superconducting charge qubits, the artificial two-level atoms, without an external biased flux [38]. The nonlinear Hamiltonian for three-wave mixing is derived by quantum mechanical calculations through the Fröhlich-Nakajima transformation [39]. The nonlinear interaction constant depends on the frequencies of the qubit and the pump microwave field. A large nonlinear interaction strength can be obtained with current experimental techniques. The squeezing efficiency is mainly dependent on the quality factor of the superconducting TLR, which is also the case in nonlinear optical cavity experiments.

This paper is organized as follows: In the next section, the physical setup is described in detail and the effective Hamiltonian is obtained by means of the Fröhlich-Nakajima transformation. In Sec. III, the squeezing effect is analyzed with and without the noise. Finally, the main results are summarized in Sec. IV.

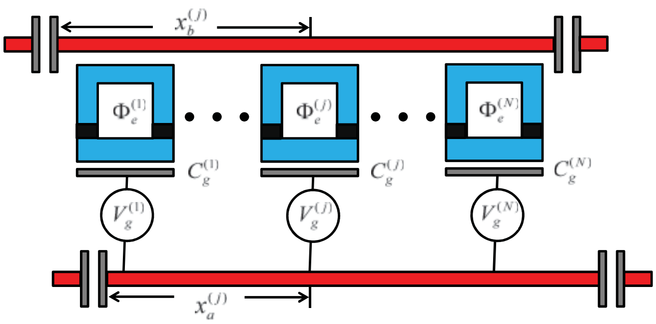

FIG. 1. (Color online) Schematic of combined system consisting of two superconducting TLRs and a series of superconducting charge qubits. The first TLR, shown at the bottom, is coupled to the charge qubits through the gate capacitances $C_g^{(j)}$. The second TLR, shown at the top, is coupled to the charge qubits through magnetic field induced by its quantized current. Here, the jth qubit is placed at $x_a^{(j)}$ with respect to the first TLR, where the quantized voltage is maximum and the quantized current is zero. Meanwhile, it is also placed at $x_b^{(j)}$ with respect to the second TLR, where the quantized current reaches a maximum and the quantized voltage vanishes.

II. MODEL

Our proposed circuit QED system is schematically illustrated in Fig. 1. A series of superconducting charge qubits are fabricated inside two one-dimensional superconducting TLRs. In the charge representation, the Hamiltonian of a single superconducting charge qubit is [40]

$$H_q = -\frac{1}{2}E_C(1-2n_g)\sigma_z - E_J\cos\left(\pi\frac{\Phi_e}{\Phi_0}\right)\sigma_x, \tag{1}$$

where

$$E_C = \frac{2e^2}{C_\Sigma} \tag{2}$$

is the single Cooper-pair charging energy with a total capacitance $C_\Sigma = 2C_J + C_g$, the Josephson junction capacitance is C_J, the gate capacitance is C_g, E_J is the Josephson coupling energy,

$$n_g = \frac{C_g V_g}{2e} \tag{3}$$

is the gate charge induced by the gate voltage V_g, and Φ_e is the externally applied flux. For a charge qubit located inside the first TLR, the quantized voltage of the first TLR also applies to the gate capacitance C_g of the charge qubit. The coupling between the qubit and the second TLR comes from the quantized flux, which is induced by the quantized current in the second TLR, through the effective area s of the superconducting quantum interference device (SQUID) configuration. At the antinodes, the quantized voltage and current in the first and second TLR take their maximum amplitudes [25]:

$$V_q = \sum_k V_0^{(k)}(a_k^\dagger + a_k), \tag{4}$$

$$I_q = -i\sum_k I_0^{(k)}(b_k^\dagger - b_k), \tag{5}$$

respectively, with

$$V_0^{(k)} = \sqrt{\frac{\hbar\omega_k^{(1)}}{L_1 c_1}}, \tag{6}$$

$$I_0^{(k)} = \sqrt{\frac{\hbar\omega_k^{(2)}}{L_2 l_2}}. \tag{7}$$

Here,

$$\omega_k^{(i)} = \frac{k\pi}{L_i\sqrt{l_i c_i}} \tag{8}$$

is the frequency of the kth mode in the ith TLR ($i = 1, 2$) with L_i, l_i, and c_i being the length, the inductance, and the capacitance per unit length, respectively. The quantized flux induced by the quantized current in the second TLR is

$$\Phi_q = \frac{\mu_0 s I_q}{2\pi d}, \tag{9}$$

with d being the distance between the qubit and the second TLR. At a sufficiently low temperature, there is only one mode for each TLR, say $\omega_k^{(1)} = \omega_a$ and $\omega_k^{(2)} = \omega_b$, that couples to the qubit. Located at the point

$$x_a = \frac{n_a\pi}{\omega_a\sqrt{l_1 c_1}} \quad (n_a = 0,1,\ldots,k_a), \tag{10}$$

where the quantized voltage

$$V_q = V_0(a^\dagger + a) \tag{11}$$

in the first TLR is maximum, the qubit is only capacitively coupled to mode a for the quantized current and thus the flux vanishes. Similarly, since the qubit is simultaneously placed at the point

$$x_b = \frac{\left(n_b + \frac{1}{2}\right)\pi}{\omega_b\sqrt{l_2 c_2}} \quad (n_b = 0,1,\ldots,k_b), \tag{12}$$

with the maximum quantized current in the second TLR, it is only inductively coupled to mode b with quantized flux

$$\Phi_q = -i\phi_b(b^\dagger - b) \tag{13}$$

applied to the SQUID's effective area because of the vanishing quantized voltage. The Hamiltonian of the joint system reads ($\hbar = 1$)

$$H_1 = \omega_a a^\dagger a + \omega_b b^\dagger b - \frac{1}{2}E_c(1-2n_g)\sigma_z - E_J\cos\frac{\pi(\Phi_e+\Phi_q)}{\Phi_0}\sigma_x + \frac{eC_g V_0}{C_\Sigma}(a^\dagger + a)\sigma_z, \tag{14}$$

where Φ_e and $\Phi_0 = h/(2e)$ are respectively the external biased flux and the flux quantum. Besides, because mode a is capacitively coupled to the qubit, the polarization of its electric field is in the coplane of the TLR. And the polarization of the magnetic field of mode b is perpendicular to the plane since it interacts with the qubit by the magnetic flux through the loop enclosed by the qubit.

We set the external biased flux at the special value $\Phi_e = 0$; that is, the external biased flux is turned off. To second order in Φ_q/Φ_0, the effective Josephson coupling energy is $-E_J[1-\pi^2\Phi_q^2/(2\Phi_0^2)]$, which has a quadrature dependence on Φ_q. Now we choose the eigenenergy basis of the qubit; namely,

$$|g\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle, \tag{15}$$

$$|e\rangle = \cos\left(\frac{\theta}{2}\right)|1\rangle - \sin\left(\frac{\theta}{2}\right)|0\rangle \tag{16}$$

to simplify the total Hamiltonian in Eq. (14). Here,

$$\theta = \tan^{-1}\left[\frac{2E_J}{E_c(1-2n_g)}\right] \tag{17}$$

is the mixing angle. Under the rotating-wave approximation, the Hamiltonian in Eq. (14) can be rewritten as

$$H_2 = \omega_a a^\dagger a + \omega_b b^\dagger b - \tfrac{1}{2}\Omega\rho_z + g_a a^\dagger \rho_- + g_b b^{\dagger 2}\rho_- + \text{H.c.}, \tag{18}$$

where

$$\Omega = \sqrt{E_c^2(1-2n_g)^2 + 4E_J^2} \tag{19}$$

is the transition frequency of the qubit,

$$g_a = -\frac{eC_g V_0}{C_\Sigma}\sin\theta \tag{20}$$

and

$$g_b = -\frac{E_J\pi^2\phi_b^2}{2\Phi_0^2}\cos\theta \tag{21}$$

are the coupling coefficients between the qubit and the first and second TLRs, respectively, and

$$\rho_z = |g\rangle\langle g| - |e\rangle\langle e|, \tag{22}$$

$$\rho_+ = \rho_-^\dagger = |e\rangle\langle g|. \tag{23}$$

In our circuit-QED system, the distance between two arbitrary neighboring qubits is set large enough so that there is no direct interaction between them, and each qubit is placed at the points where the amplitudes of the quantized voltage of the first TLR and the quantized current of the second TLR take their maximum values [41]. Then, the total Hamiltonian of our considered system reads

$$H_t = \omega_a a^\dagger a + \omega_b b^\dagger b - \frac{1}{2}\Omega\sum_{j=1}^{N}\rho_z^{(j)} + \sum_{j=1}^{N}\left(g_a^{(j)} a^\dagger \rho_-^{(j)} + g_b^{(j)} b^{\dagger 2}\rho_-^{(j)} + \text{H.c.}\right), \tag{24}$$

where for simplicity we have assumed that the transition frequencies of all qubits are the same. In the above Hamiltonian, we have used the rotating-wave approximation. Under certain conditions, the counter-rotating terms may lead to observable effects [42]. In order to get the effective interaction Hamiltonian of the two TLRs, we have to eliminate the degrees of freedom of the qubits. Here, we adopt a canonical transformation—the Fröhlich-Nakajima transformation [39,42,43], which has been widely used in condensed matter physics—to eliminate the variables of qubits.

Our system works in the large-detuning regime; that is,

$$|\Delta_a| \equiv |\Omega - \omega_a| \gg \sqrt{\sum_j g_a^{(j)2}}, \tag{25}$$

$$|\Delta_b| \equiv |\Omega - 2\omega_b| \gg \sqrt{\sum_j g_b^{(j)2}}. \tag{26}$$

In this regime, the two TLRs do not change the populations of the qubits but only result in Stark shifts in the qubits' energy levels [44–46]. Therefore, the effective interaction Hamiltonian of the two TLRs can be obtained by keeping all of the qubits in their ground states, which can be easily implemented by current experimental techniques [40,41]. Applying a unitary transformation

$$U = \exp\left[\sum_j\left(\frac{g_a^{(j)}}{\Delta_a}a^\dagger\rho_-^{(j)} + \frac{g_b^{(j)}}{\Delta_b}b^{\dagger 2}\rho_-^{(j)} - \text{H.c.}\right)\right] \tag{27}$$

to the Hamiltonian in Eq. (24), we obtain an effective Hamiltonian

$$H \simeq \omega_a a^\dagger a + \omega_b b^\dagger b - \kappa(ab^{\dagger 2} + a^\dagger b^2), \tag{28}$$

where

$$\kappa = -\sum_j g_a^{(j)} g_b^{(j)}\frac{\Delta_a + \Delta_b}{2\Delta_a\Delta_b} \tag{29}$$

is the nonlinear coupling constant between the two TLRs. In the case of

$$\omega_a = 2\omega_b = 2\omega, \tag{30}$$

in the interaction picture, the above Hamiltonian becomes

$$H_I = \kappa(ab^{\dagger 2} + a^\dagger b^2). \tag{31}$$

III. SQUEEZING

In the parametric approximation, the pump microwave field is treated classically and pump depletion is neglected. The Hamiltonian in Eq. (31) becomes

$$\mathscr{V} = \kappa\beta(b^{\dagger 2}e^{-i\varphi} + b^2 e^{i\varphi}), \tag{32}$$

where β and φ are the real amplitude and phase of the coherent pump microwave field. The evolution operator on the state of the second TLR is

$$S(\xi) = \exp\left[-i\frac{\xi}{2}(b^{\dagger 2}e^{-i\varphi} + b^2 e^{i\varphi})\right], \tag{33}$$

where

$$\xi = \Omega_p t, \tag{34}$$

and $\Omega_p = 2\kappa\beta$ is the effective Rabi frequency. This is a squeezing operator on the second TLR with squeezing parameter ξ. For the phase of the coherent pump microwave field $\varphi = \pi/2$, it can be calculated directly by using the transformation

$$S^\dagger(\xi) b S(\xi) = b\cosh\xi - b^\dagger\sinh\xi, \tag{35}$$

where the variance in one of the quadratures

$$X_1 = \tfrac{1}{2}(b^\dagger + b) \tag{36}$$

decreases exponentially:

$$\Delta X_1 = \sqrt{\langle X_1^2 \rangle - \langle X_1 \rangle^2} = \Delta X_1(0) e^{-\xi}. \tag{37}$$

In order to obtain a better squeezing efficiency, one would like a larger nonlinear interaction coupling. For the experimentally realizable parameters $\omega/(2\pi) = 10$ GHz, $\Omega/(2\pi) = 10^{10}$ to 10^3 Hz, $E_J/(2\pi) = 4$ GHz, $s/d = 20\ \mu$m, $V_0 = 2\ \mu$V, $C_g/C_\Sigma = 0.1$, $\phi_b/\Phi_0 = 5 \times 10^{-5}$, $\beta = 10$, and $N = 5$, the nonlinear coupling constant $\kappa/(2\pi)$ is about 5.72 MHz and the effective Rabi frequency $\Omega_p/(2\pi)$ is about 114.4 MHz. Here, for the sake of a sufficiently large κ, we tune the level spacing of the qubits Ω very close to ω (i.e., the frequency of mode b), which can be feasibly achieved by adjusting the gate voltage $V_g^{(j)}$ applied on the qubits.

Following the standard quantum theory of damping, we investigate the influence of the decoherence of the system on the squeezing efficiency. In order to include the influence of the qubits, we use the master equation [1]

$$\begin{aligned}\frac{d\varrho}{dt} &= -\frac{i}{\hbar}[\mathscr{H},\varrho] + \frac{1}{2}\gamma_n(2b\varrho b^\dagger - b^\dagger b\varrho - \varrho b^\dagger b)\\ &\quad + \frac{1}{2}\sum_j \gamma_q^{(j)}(2\rho_-^{(j)}\varrho\rho_+^{(j)} - \rho_+^{(j)}\rho_-^{(j)}\varrho - \varrho\rho_+^{(j)}\rho_-^{(j)})\\ &\quad + \sum_j \gamma_\varphi^{(j)}\big(\rho_z^{(j)}\varrho\rho_z^{(j)} - \varrho\big),\end{aligned} \tag{38}$$

where ϱ is the density matrix of the combined system of the qubits and the second TLR,

$$\gamma_n = \frac{\omega}{Q} \tag{39}$$

and Q are the the decay rate and quality factor, respectively, of the second TLR, and $\gamma_q^{(j)}$ and $\gamma_\varphi^{(j)}$ are the relaxation rate and dephasing rate of the qubits, respectively. The Hamiltonian form is chosen as

$$\mathscr{H} = \frac{\kappa\beta}{N}(b^{\dagger 2}e^{-i\varphi} + b^2 e^{i\varphi})\sum_j \rho_z^{(j)} \tag{40}$$

to include all sources of decoherence and dissipation of the qubits and TLR. For convenience of computation, we assume all $\gamma_q^{(j)}$ and $\gamma_\varphi^{(j)}$ are equal, and Eq. (38) becomes

$$\begin{aligned}\frac{d\varrho}{dt} &= -\frac{i}{\hbar}[\mathscr{H},\varrho] + \frac{1}{2}\gamma_n(2b\varrho b^\dagger - b^\dagger b\varrho - \varrho b^\dagger b)\\ &\quad + \frac{\gamma_q}{2}\sum_j(2\rho_-^{(j)}\varrho\rho_+^{(j)} - \rho_+^{(j)}\rho_-^{(j)}\varrho - \varrho\rho_+^{(j)}\rho_-^{(j)})\\ &\quad + \gamma_\varphi\sum_j\big(\rho_z^{(j)}\varrho\rho_z^{(j)} - \varrho\big).\end{aligned} \tag{41}$$

In the numerical simulation, we truncate the number of levels in mode b as high as 256. Moreover, in order to guarantee its validity, we double the truncation number and find that the result converges. In Fig. 2, we plot the time dependence of the variance $\Delta X_1(t)/\Delta X_1(0)$. We have chosen the following conservative experimental parameters [12,13,20]: $\gamma_q = 1$ MHz, $\gamma_\varphi = 10$ MHz, and $Q = 5 \times 10^5$. At the temperature $T = 20$ mK, the initial state of the second TLR is in a thermal equilibrium state and the qubits are in their ground states. Furthermore, since the effective coupling between the qubit and mode a is enhanced by a factor β, which is the square root of the average photon number in mode a, it requires $|\Delta_a| \gg \beta g_a^{(j)}$ in order for the drive not to modify the population distribution in the qubit [47]. It has been numerically confirmed that, with the parameters mentioned above, the derivation from all population in the ground state is no more than 2%. As a consequence, the approximation of all qubits in the ground state is valid.

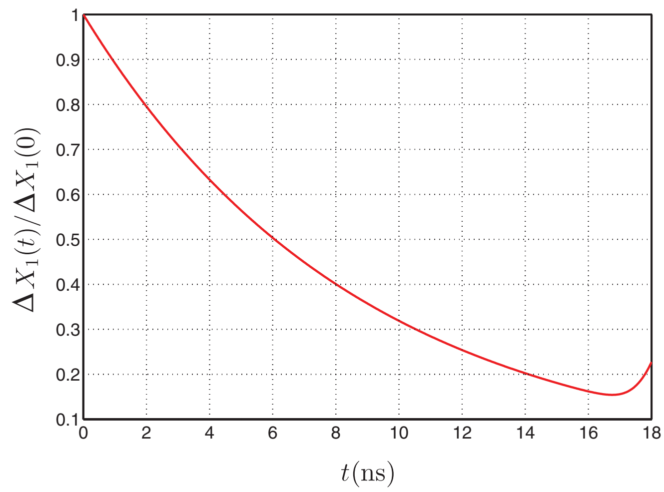

FIG. 2. (Color online) Time dependence of $\Delta X_1(t)/\Delta X_1(0)$. There is a minimum due to the fluctuation in the system.

Due to the influence of noise in the system, there is a minimum in $\Delta X_1(t)/\Delta X_1(0)$. This means the dissipation and decoherence may corrupt the squeezing effect severely. The squeezing effect may be destroyed eventually when the squeezing time is long enough. In order to investigate what is the main factor that affects the squeezing efficiency, we numerically calculate the dependence of $\Delta X_{1\mathrm{Min}}/\Delta X_1(0)$ on γ_q, γ_φ, and γ_n. However, we find that the minimum of ΔX_1 does not vary when γ_q and γ_φ are changed. It is easy to understand that the qubits are always kept in their ground states and play the role of the nonlinear medium in the squeezing process. In Fig. 3, we plot the minimum of ΔX_1 versus the

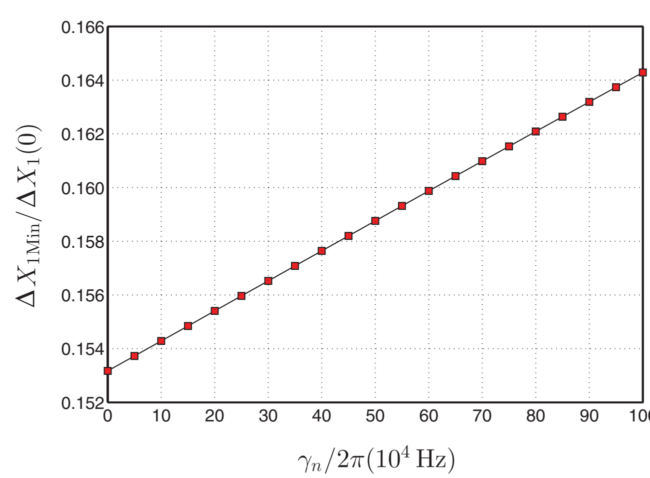

FIG. 3. (Color online) Minimum of $\Delta X_1/\Delta X_1(0)$ versus the decay rate γ_n of the second TLR. The red squares are obtained by numerically solving Eq. (41) with $\gamma_q = 1$ MHz and $\gamma_\varphi = 10$ MHz.

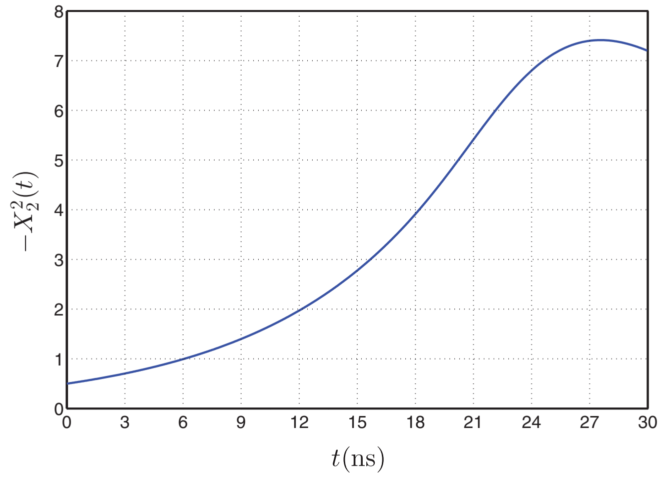

FIG. 4. (Color online) Time dependence of $-X_2^2(t)$. Although it increases with time at the early stage due to the uncertainty relation, there is a maximum which ensures the limited expansion of Josephson energy.

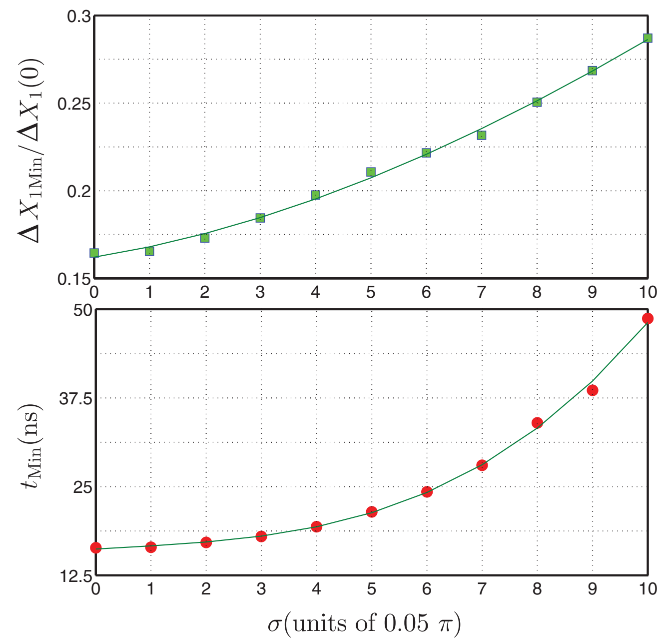

FIG. 5. (Color online) Minimum of ΔX_1 (green squares) and the time t to reach the minimum (red circles) for different σ.

decay rate of the second TLR. As expected, the minimum of ΔX_1 increases with decreasing TLR quality factor.

In obtaining the Hamiltonian (18), we approximated the Josephson energy to second order in Φ_q/Φ_0, which is proportional to $X_2 = i(b^\dagger - b)/2$. Since there is the uncertainty relation governing the time evolution of both X_1 and X_2 (i.e., $\Delta X_1 \Delta X_2 \geqslant 1/2$), it is reasonable to question the validity of the limited expansion given above. However, as shown in Fig. 4, there is a finite maximum for the time evolution of $-\Delta X_2^2$ due to the nonvanishing minimum of ΔX_1. Therefore, it is both the finite maximum of $-\Delta X_2^2$ and the small coefficient $\phi_b/\Phi_0 = 5\times 10^{-5}$ that ensure the validity of the approximation.

The superconducting charge qubits, playing the role of the nonlinear medium, are kept in their ground states, and we do not apply any operations or measurements on the them in the squeezing process. As a result, the decoherence of the qubits does not affect the squeezing efficiency, which has been verified by our numerical solution of Eq. (41). In order to obtain a large squeezing efficiency, we just need to improve the quality factor of the TLR. Our numerical results in Fig. 2 are similar to the analytical results of Ref. [48] in which the initial state of the signal mode is chosen as the vacuum state.

On the other hand, in the above calculation, we have assumed a fixed phase of the driving field, (i.e., $\varphi = \pi/2$). However, in a realistic experiment, the phase may be a random number distributed around some mean due to the noise. In order to take this factor into consideration, we apply the quantum Monte Carlo method [49] to the stochastic process, in which φ is normally distributed around $\pi/2$ with variance σ^2. As shown in Fig. 5, the squeezing effect still emerges, although the minimum of ΔX_1 increases as the variance increases. Meanwhile, the time to reach the minimum is significantly delayed. This observation is consistent with the previous study [50], which clarified the role of phase noise in the cooling process of nanomechanical resonators [51].

In the above discussions, we have proposed an experimental setup for the generation of the squeezed state in the TLR. In order to verify our scheme, it is necessary to detect the squeezed state mentioned above after it is generated. Recently, a two-mode squeezed state has been observed in the microwave-frequency domain [52]. Enlightened by this progress, we provide a detection scheme for the generated squeezed state. The above signal is input into a high-electron-mobility transistor (HEMT). Then, the signal is mixed with a local oscillator tone at the same frequency as mode b. Every few ns, the voltages are digitized with an analog-to-digital converter. By means of a field programable gate array, we digitally filter the data with a sinc Chebyshev filter function $f(\Delta)$. Therefore, microwave photons with frequency in this window are linearly detected. Afterwards, two quadrature components are extracted from a digital data processing about the results of two operators (i.e., $x + ip = b + ih^\dagger$ and $x - ip = b^\dagger - ih$). Here, $b_{\mathrm{out}} = \int_{-\infty}^{\infty} d\Delta f(\Delta) b(\Delta)$ and h is the noise annihilation operator mainly due to the HEMT amplifier noise. Finally, the noise can be extracted with the relation $\langle X_1^2\rangle = \langle x^2\rangle_{\mathrm{on}} - \langle x^2\rangle_{\mathrm{off}} + 1/4$, where the subscripts "on" and "off" refer to mode b being on or off. Besides, another scheme based on a microwave quantum homodyne measurement technique is also competent for measuring the variance ΔX_1 [53].

IV. DISCUSSIONS AND CONCLUSION

In summary, we have studied the theory of degenerate three-wave mixing by using circuit quantum electrodynamics in superconducting solid state circuits. We derived the Hamiltonian for degenerate three-wave mixing which seemed to be given phenomenologically in text books for quantum optics. Therefore, we provide a possible microscopic model for the three-wave mixing. Our results may be helpful for understanding degenerate three-wave mixing in quantum optics. The nonlinear coupling constant κ is controllable and dependent on the frequencies of the TLRs and the qubits. We investigated the generation of squeezed state using three-wave mixing. The main obstacle preventing large squeezing efficiency is the decay rate of the TLR. Since the qubits are kept in their ground states and play the role of a nonlinear medium, their decoherence does not affect the squeezing efficiency. Furthermore, due to the $\sqrt{N}$-enhanced coupling between the field and the qubits, our scheme requires much less time to reach the maximum squeezed state with comparison to the scheme with only one qubit [31].

Note added in proof. Recently, we have learned that the microscopic theory of nonlinear polarizabilities already exists in books concerning nonlinear optics (see, e.g., Ref. [54]).

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