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A Scheme for Simulation of Quantum Gates by Abelian Anyons*

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Abstract *Anyons can be used to realize quantum computation, because they are two-level systems in two dimensions. In this paper, we propose a scheme to simulate single-qubit gates and CNOT gate using Abelian anyons in the Kitaev model. Two pairs of anyons (six spins) are used to realize single-qubit gates, while ten spins are needed for the CNOT gate. Based on these quantum gates, we show how to realize the Grover algorithm in a two-qubit system.*

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Key words: fractional statistics, anyon, quantum gate

1 Introduction

Anyons are exotic quasi-particles in two dimensions, of which the statistical characteristics are quite different from those of bosons and fermions. The model of anyons, which obeys fractional statistics, was put forward by Wilczek.[1−4] However, anyons are not only a theoretical model but they can be also experimentally detected, i.e., the fractional quantum Hall effect.[4−7] They are two-level systems with the ground state and the excited state. When we braid two different kinds of anyons, there will be an additional phase factor in the wave function, which is determined by the winding number and the statistical parameter. Moreover, anyons can only be created in pairs. Due to their special characteristics, they have been applied to many fields. For instance, high-T_c superconductivity[8] and fault-tolerant topological quantum computing.[9−10] The study of anyons and Gentile statistics particles has attracted lots of interests recently.[11−21]

So far the models of anyons are concerned, there is a famous spin lattice model proposed by Kitaev, namely the first Kitaev model.[22−23] It is not only exactly solvable in theory but can also be realized in experiment. In this model, there is a spin on each edge, as shown in Fig. 1. The ground state and the excited state are the two relevant states in this model. The excited particles are created in pairs. There are four super-selection sectors in the Kitaev model, i.e., 1 (the vacuum), e (electric charge), m (magnetic vortices) and $\varepsilon = e \times m$. When a Pauli-Z operation is applied to an edge spin, particles e are created on the two vertices linked to the spin. When a Pauli-X operation is performed on an edge spin, particles m are created in the two connected plaquettes. Furthermore, when two different particles are braided with each other, the additional phase is generated. For example, when a particle m goes around a particle e for one circle, the wave function possesses a π phase. Since the statistical parameter is 1/2, they are called 1/2-anyons as well.

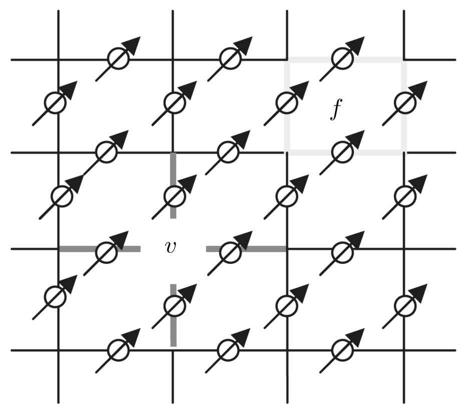

Fig. 1 The first Kitaev spin lattice model with spins located in the center of an edge. Anyons are excited on the vertices and faces. Here, the vertex (v) is the cross of four edges labeled in red. And the face (f) is enclosed by four edges labeled in green.

As shown in Fig. 2, six spins are enough to show the braiding operation of anyons.[24−25] The statistics of anyons was displayed in photonic quantum simulator[24] and cold atoms controlled by dynamic laser[25] as well. Both of these experiments demonstrated that the phase factor of anyons in the braiding operation can be detected. In this paper, in the light of Refs. [24–25], we use Abelian anyons in the Kitaev model to realize quantum gates. Furthermore, in virtue of these gates, we propose a scheme to implement the Grover algorithm.[26]

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This paper is organized as follows. In Sec. 2, we simulate single-qubit gates using Abelian anyons in the Kitaev model. In Sec. 3, we show how to put the CNOT gate into practice by the same model. Then, in Sec. 4, by virtue of the above gates, we present a scheme for realizing the Grover algorithm. At the end of this paper, a brief summary is concluded in Sec. 5.

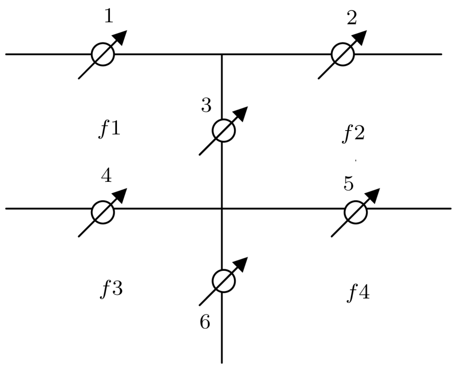

Fig. 2 Six-spin Kitaev model taken from Ref. [24]. There are six spins labeled from 1 to 6 and four faces labeled as $f1$, $f2$, $f3$, and $f4$, respectively.

2 Single Qubit Quantum Gates

As shown in Fig. 1, anyons are excited on the vertices and faces, and they can only be generated in pairs.[22−23] The Hamiltonian of the first Kitaev model is[22−23]

$$H = -\sum_{\text{vertices}} A_v - \sum_{\text{plaquettes}} B_f\,, \tag{1}$$

where

$$A_v = \prod_{j\in\,\text{star}\,(v)} X_j\,, \tag{2}$$

$$B_f = \prod_{j\in\,\text{plaquettes}\,(f)} Z_j \tag{3}$$

are stabilizer operators, which commute with each other. Here, X_j, Y_j, and Z_j are the Pauli operators for j-th boundary spin. Both A_v and B_f have two eigenvalues with $+1(-1)$ corresponding to the ground (excited) state, respectively. As shown in Fig. 2, six spins are the minimum number of spins to demonstrate the braiding statistics of anyons.[24−25] The Hamiltonian H_6 and its ground state $|\psi_6\rangle$ for the case with six spins have been given in Ref. [24]. Here, we also utilize six spins (four anyons) in order to simulate single-qubit quantum gates. And we use $|0\rangle$ $(= |\psi_6\rangle)$ and $|1\rangle$ to denote the ground state and the excited state, respectively. For the ground state, there is no particle created. For the excited state, a pair of e particles can be created on the two vertices connected with spin 3. There are two kinds of basic operations in need. One is the braiding of anyons, while the other is $\sqrt{Z}$ rotation.

To show the effect of the $\sqrt{Z}$ operation explicitly, we plot Fig. 3. It can be considered as a rotating in the Hilbert space spanned by the two states $|0\rangle$ and $|1\rangle$. When $\sqrt{Z}$ rotation is performed on the ground state $|0\rangle$, it becomes $(|0\rangle + |1\rangle)/\sqrt{2}$. When we apply it to the excited state $|1\rangle$, we obtain a different superposition $(-|0\rangle + |1\rangle)/\sqrt{2}$. In other words, $\sqrt{Z}$ operation is a rotation of 45° counterclockwise. For the braiding operation, when we braid two different kinds of anyons, the wave function attains an additional phase factor $\exp(\mathrm{i}2\pi k\alpha)$ with k being the winding number and α being the statistical parameter.[6−7] In contrast, if two anyons of the same kind are braided, there is no such phase created.

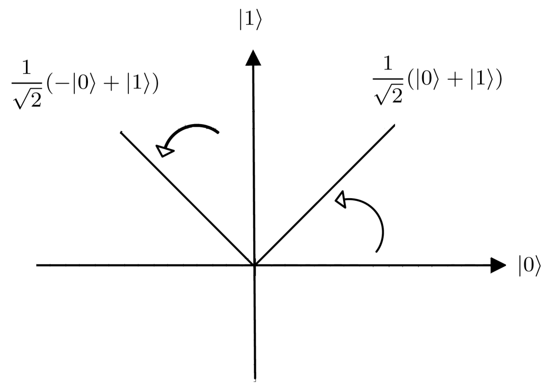

Fig. 3 The effect of $\sqrt{Z}$ rotation.

2.1 *Pauli-Z Gate*

In order to simulate single-qubit operations in the Kitaev model, there are six spins needed as shown in Fig. 2. First of all, we introduce the steps for Pauli-Z gate since it is the simplest case in our method. We only need two steps to simulate it.

(i) Perform X operation on spin 4.

(ii) Apply three X operations on spins 6, 5, and 3 in turn.

In the above procedures, X, Y, and Z are the Pauli operations for the single spin. In step one, a pair of m particles are generated on plaquettes $f1$ and $f3$. In step two, when an X operation is applied to spin 6, there are two m particles created on $f3$ and $f4$. Due to the fusion law, two m particles created on the same vertex cancel.[22] Afterwards, the same operation is performed on spins 5 and 3 sequentially.

The effect of step two will be explained explicitly as follows. If the state is initially the ground state $|0\rangle$, it will remain at the ground state without an additional phase since there are no e particles on the both vertices connected to spin 3. In contrast, if the state is initially prepared in the excited state $|1\rangle$, it will stay at the same state with an extra phase factor $\exp(\mathrm{i}2\pi \times 1/2) = -1$ for the sake of two e particles on the both vertices linked to spin 3.

In all, a Pauli-Z gate is realized by means of the above two steps.

2.2 *Pauli-X Gate*

In combination with an additional step, we will show how to simulate Pauli-X gate as follows.

(i) Perform X operation on spin 4.

(ii) Apply three X operations to spins 6, 5, and 3 in turn.

(iii) Operate Z on spin 3.

As stated in the previous subsection, the first two steps brings a Pauli-Z gate into being. Then, in step three, when a Z operation is applied to spin 3, the states $|0\rangle$ and $-|1\rangle$ become $|1\rangle$ and $|0\rangle$, respectively. As a result, a Pauli-X gate is completed.

Besides, by making use of a Pauli-X and Pauli-Z gates, we can achieve a Pauli-Y gate. Furthermore, based on the Pauli gates, a Hadamard gate can be implemented if we utilize a $\sqrt{Z}$ rotation after a Pauli-Z gate.

2.3 *Complement*

As shown in the previous subsections, we give these four single-qubit quantum gates using Abelian 1/2-anyons in the Kitaev model. If the Abelian anyons which we use is not Abelian 1/2-anyons, we can simulate more single-qubit quantum gates, e.g., the phase gate. Using the steps by which we simulate the Pauli-Z gate, we can create a phase gate. The difference is that the phase factor is not -1 any longer but $\exp(\mathrm{i}2\pi k\alpha)$ when we braid different kinds of anyons. Thus, single-qubit quantum gates can be simulated by Abelian anyons.

3 CNOT Gate

In this part, we simulate two-qubit CNOT gate by means of Abelian 1/2-anyons in the Kitaev model. Because six spins are enough to show the braiding operation, we need ten spins at least to bring the CNOT gate into being. As shown in Fig. 4, the red part is the control qubit in state $|a\rangle$ ($a = 0, 1$), while the black part is the target qubit in state $|b\rangle$ ($b = 0, 1$). We need three steps to simulate the CNOT gate.

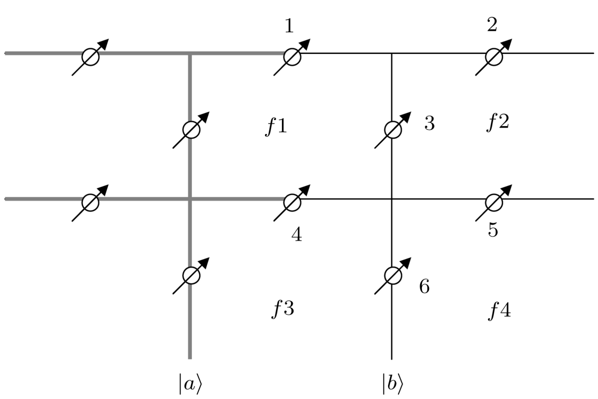

Fig. 4 The simulation of the CNOT gate. The red part is the control qubit in state $|a\rangle$ with $a = 0, 1$, while the black part is the target qubit in state $|b\rangle$ with $b = 0, 1$.

(i) Apply X operation on spin 4.

(ii) Perform X operations on spins 6, 5, 3, 4 for $a + 2$ circles.

(iii) Use Z^a operation on spin 3.

When the state of the control qubit is the ground state $|0\rangle$, we have $a = 0$. In step one, two m particles are created on plaquettes $f1$ and $f3$. In step two, we use seven X operations on spins 6, 5, 3, 4, 6, 5, and 3 sequentially. In this step, since we braid around the vertex under spin 3 for two circles, there will be no particles and thus extra phase produced. No matter what the state of the target qubit is, there is no change for its state. In the next procedure, because $Z^0 = 1$, the target qubit remains itself.

Now, we consider the control qubit to be in the excited state $|1\rangle$, namely $a = 1$. Step one also creates two m particles on plaquettes $f1$ and $f3$. In step two, we perform eleven X operations on spins 6, 5, 3, 4, 6, 5, 3, 4, 6, 5, and 3, one by one. Obviously, we braid around the vertex under spin 3 for three circles. If the state of the target qubit is $|0\rangle$, it remains $|0\rangle$ after step two. Otherwise, it becomes $-|1\rangle$. Through a Z rotation in step three, we transform $|0\rangle$ and $-|1\rangle$ into $|1\rangle$ and $|0\rangle$, respectively. Thus, the CNOT gate is finished.

4 Grover Algorithm

Grover algorithm is one of the most famous algorithms in quantum computing.[26] It is a searching algorithm over an unsorted database, and its potential wide applications in science and engineering problems.[27] It also plays a role in quantum communication such as quantum secret sharing.[28–29]

The Grover algorithm consists of four steps: i) Inverse of the marked state; ii) A Hadamard–Walsh transformation; iii) Inversion of the $|0\rangle$ state and iv) Another Hadamard–Walsh transformation. The success rate of this original Grover algorithm is not 100%. An exact Grover algorithm was constructed,[30] using the phase matching condition.[31–32]

We use a 2-qubit case as an example. The two-qubit case Grover algorithm is special because the Grover algorithm also finds the marked state with full probability. Though this fact has been known more than 10 years ago, its rigorous proof was given only recently by using very advanced mathematical tools.[33]

The state of the 2-qubit system can be written as

$$|\psi\rangle = \sqrt{\frac{1}{4}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \,. \tag{4}$$

We assume to find the marked state $|11\rangle$. As shown in Fig. 2, in Kitaev model, $|00\rangle$ is a blanket state. When a Pauli-Z operator acts on spin 3, two e particles are created on the both vertices connected to that spin, and this case is the state $|01\rangle$. When a Pauli-X operator acts on spin 4, two m particles are created on $f1$ and $f3$, the state becomes $|10\rangle$. $|11\rangle$ is the state which has two e particles on two vertices connected to spin 3 and two m particles on $f1$ and $f3$. Here we have $|\alpha\rangle = |11\rangle$ and

$$|\beta\rangle = \sqrt{\frac{1}{3}}(|00\rangle + |01\rangle + |10\rangle)\,. \tag{5}$$

The four steps of one round of search process can be realized as what follows.

(i) The inversion operation I_α can be written as $I_\alpha = I - 2|\alpha\rangle\langle\alpha|$. To realize this operation, we perform four X operators on spins 6, 5, 3, and 4 in turn. After these operations, the marked state $|\alpha\rangle$ becomes $-|\alpha\rangle$. Those four X operators constitute a braiding operation. In the braiding operation, an m particle goes around an e particle, the process creates a phase -1.

(ii) Perform Hadamard–Walsh gate on single qubit $|0\rangle$ and $|1\rangle$. The Hadamard–Walsh gate on a single qubit produces the following change,

$$H|0\rangle = \sqrt{\frac{1}{2}}(|0\rangle + |1\rangle)\,, \tag{6}$$

$$H|1\rangle = \sqrt{\frac{1}{2}}(|0\rangle - |1\rangle)\,. \tag{7}$$

(iii) The inversion operation $I_0 = I - 2|0\rangle\langle 0|$. Inverts the sign of $|00\rangle$ component. To realize this operation, we first perform one Z operator on spin 3. Then we use four X operators on spin 4, 6, 5, and 3 in turn. At last, Z operator is performed on spin 3 again. After using these operations, $|00\rangle$ becomes $-|00\rangle$ and other states remain themselves.

(iv) Use Hadamard–Walsh gate again.

It will find the marked state $|\alpha\rangle$ after using these four steps for only one round. It is interesting to note that the two inversion operations contain the essential part of a CNOT gate, and it is to find out this by comparing the operations in step 1 and step 3 in the Grover algorithm and that in the CNOT.

5 Conclusion

In this paper, by making use of Abelian anyons in the Kitaev model, we propose a scheme to simulate single-qubit gates and the CNOT gate. We establish a relation between anyons and quantum information. In the light of Refs. [24–25], we use six-spin Kitaev model to simulate single-qubit gates. Besides, ten-spin Kitaev model is utilized to simulate the CNOT gate. Based on the above quantum gates, we can also put forward a proposal for realizing the Grover algorithm in the Kitaev spin lattice model.

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