

Induced entanglement enhanced by quantum criticality

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Two-qubit entanglement can be induced by a quantum data bus interacting with them. In this paper, with the quantum spin chain in the transverse field as an illustration of the quantum data bus, we show that such induced entanglement can be enhanced by the quantum phase transition (QPT) of the quantum data bus. We consider two external spins simultaneously coupled to a transverse field Ising chain. By adiabatically eliminating the degrees of the chain, the effective coupling between these two spins is obtained. The matrix elements of the effective Hamiltonian are expressed in terms of the dynamical structure factor (DSF) of the chain. The DSF is the Fourier transformation of the Green function of an Ising chain and can be calculated numerically by a method introduced by Derzhko and Krokhnalskii [Phys. Rev. B **56**, 11659 (1997)]. Since all characteristics of QPT are embodied in the DSF, the dynamical evolution of the two external spins displays singularity in the vicinity of the critical point.

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I. INTRODUCTION

Entanglement lies at the heart of quantum mechanics and thus can be regarded as a resource for quantum information processing. In recent years, many people have demonstrated that quantum entanglement can offer an intrinsic clarification of quantum criticality of a many-body system. For example, in Refs. [1,2] it was proven that the derivative of the nearest-neighbor entanglement diverged at the critical point. Furthermore, one of the authors (C.P.S.) and his collaborators studied the dynamical ultrasensitivity of the induced quantum critical system, which concerned the excited states as well as the ground state [3,4]. Usually, the former has not been discussed in the investigation for the above-mentioned intrinsic entanglement in many-body systems.

In this paper, we consider the quantum entanglement of the many-body problem in the viewpoint of quantum information processing, whether or not quantum criticality of the many-body system (as a quantum data bus) can enhance the entanglement of the external qubits interacting with the quantum data bus. Actually, the phase transition indeed creates some entanglement. A case in point is the superconducting phenomenon [5,6]. As shown in the BCS theory [7,8], a Cooper pair is created when a conductor transits from a normal state to a superconducting state. In such a pair, two electrons form a correlated entirely over hundreds of nanometers, which is a long distance in the microscopic world. We remark that the phase transition of superconducting happens at finite temperature, but this paper will focus on the occurrence of quantum phase transition (QPT) at zero temperature.

QPT is of critical importance to the quantum statistical physics. Generally speaking, QPT takes place at zero temperature. It is the situation where only the uncertainty principle plays the major role while the fluctuation due to the finite temperature does not. As some parameter is varied, a qualitative change occurs in the ground state of a quantum many-body system due to QPT [9,10]. At the critical point, long-range correlation also develops in the ground state. This

long-range correlation intuitively exhibits greater quantum entanglement between two points. References [1,2] indeed showed the entanglement became singularly longer at the critical point. These considerations directly motivate us to study the problem.

On the other hand, the interesting point of this paper is its meaning for the detection of QPT by coupling the quantum critical system to the external detector-two qubits. And various efforts have been devoted to this field. Entanglement transfer was investigated between two external spins which were coupled to a spin chain at different sites [11]. In Ref. [3], when an external spin underwent a transition from a pure state to a mixed state, the decay of the Loschmidt echo of its coupling environment described by a transverse field Ising model (TFIM) was greatly enhanced. Others suggested two external spins to detect QPT by simultaneously coupling to an XY model environment [12,13]. The former research showed the critical phenomena of QPT with exact solvable models. However, they mainly focused on $\sigma^z\sigma^z$ interaction between the external spins and the Ising chain. Thus, by extending z - z coupling to a more general coupling form, we explore the possibility of detecting QPT by two central spins.

The rest of the paper is organized as follows. In the next section, we describe the model as two external spins coupled to a transverse field Ising chain. The calculation of the effective Hamiltonian between the two spins is outlined using Fröhlich transformation in Sec. III. The matrix elements of the effective Hamiltonian are given in terms of the dynamical structure factor (DSF). In Sec. IV, the DSF is numerically calculated and the relation between the coupling constants of two spins and the variable parameter is given. Finally, the significant results are concluded in Sec. V.

II. MODEL DESCRIPTION

Two external spins are simultaneously coupled to an environment described by a one-dimensional transverse field Ising chain. By exchanging spin angular momentum with the

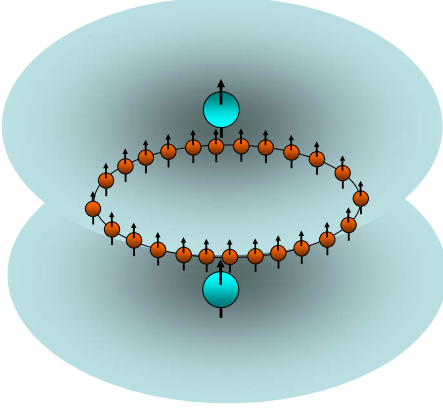


FIG. 1. (Color online) Schematic diagram of two external spins (C) simultaneously coupling to a 1D Ising chain (E).

chain, the two noninteracting spins attain effective interaction between them. To study the dynamic detail, we consider the system depicted in Fig. 1. The Hamiltonian of this model reads

$$H = H_0 + H_I = H_C + H_E + H_I, \quad (1)$$

where

$$H_C = \frac{\mu}{2}(\sigma_A^z + \sigma_B^z) \quad (2)$$

is the unperturbed Hamiltonian for the external spins,

$$H_E = - \sum_{j=1}^N (\Gamma \sigma_j^z + J \sigma_j^x \sigma_{j+1}^x) \quad (3)$$

is the Hamiltonian of the transverse field Ising model, and

$$\begin{aligned} H_I &= \sum_{\alpha=A,B} \sum_{j=1}^N \frac{J_\alpha}{\sqrt{N}} (\sigma_j^x \sigma_\alpha^x + \sigma_j^y \sigma_\alpha^y) \\ &= \sum_{\alpha=A,B} \sum_{j=1}^N \frac{2J_\alpha}{\sqrt{N}} (\sigma_j^+ \sigma_\alpha^- + \sigma_j^- \sigma_\alpha^+) \end{aligned} \quad (4)$$

is the interaction between the external spins and the environment. Here σ_α^β and σ_j^β ($\alpha=A,B, \beta=x,y,z$) are Pauli operators for the two external spins and the Ising chain, respectively, $\sigma_{\alpha,j}^\pm = (\sigma_{\alpha,j}^x \pm i\sigma_{\alpha,j}^y)/2$ are the corresponding raising and lowering operators, J_α/\sqrt{N} is the homogeneous coupling constants between the α th external spin and the j th site of the Ising chain with N being the number of sites in the chain.

First of all, H_E is diagonalized with the combination of the Jordan-Wigner transformation [14]

$$c_j = \exp\left(\pi i \sum_{k=1}^{j-1} \sigma_k^+ \sigma_k^-\right) \sigma_j^-, \quad (5)$$

and the Bogoliubov transformation [15]

$$\eta_k = \frac{1}{2} \sum_{j=1}^N [(\phi_{kj} + \psi_{kj})c_j + (\phi_{kj} - \psi_{kj})c_j^\dagger], \quad (6)$$

where for $\lambda = J/\Gamma \neq 1$,

$$\phi_{kj} = \sqrt{\frac{2}{N}} \sin(kj) \quad \text{for } k > 0, \quad (7)$$

$$\phi_{kj} = \sqrt{\frac{2}{N}} \cos(kj) \quad \text{for } k \leq 0, \quad (8)$$

$$\psi_{kj} = -\frac{1}{\Lambda_k} [(1 + \lambda \cos k)\phi_{kj} + \lambda \sin k \phi_{-kj}]. \quad (9)$$

Here,

$$2\Gamma \Lambda_k = 2\Gamma \sqrt{1 + \lambda^2 + 2\lambda \cos k} \quad (10)$$

is the energy spectrum of the quasiparticle with $k = 2\pi m/N$, $m = -N/2, \dots, N/2 - 1$ for even N , and $m = -(N-1)/2, \dots, (N-1)/2$ for odd N . For $\lambda = 1$ and $m = -N/2$,

$$\Lambda_k = 0, \quad \phi_{kj} = \sqrt{\frac{1}{N}}, \quad \psi_{kj} = \pm \sqrt{\frac{1}{N}}. \quad (11)$$

Thus, in the quasiparticle representation the Hamiltonian of the TFIM is rewritten as

$$H_E = 2\Gamma \sum_k \Lambda_k \left(\eta_k^+ \eta_k - \frac{1}{2} \right), \quad (12)$$

with the corresponding eigenstate and energy being

$$|m\rangle = \prod_k (\eta_k^+)^{n_k} |0\rangle, \quad (13)$$

$$E_m = 2\Gamma \sum_k \Lambda_k n_k - \Gamma \sum_k \Lambda_k, \quad (14)$$

respectively. Here, $|0\rangle$ is the ground state and $n_k = \eta_k^+ \eta_k$ is the particle number operator.

III. EFFECTIVE HAMILTONIAN

Generally speaking, Fröhlich transformation [16,17] is widely used in condensed matter physics. It can solve a class of problems such as the induced effective interaction between two electrons by exchanging phonons with the crystal lattice. In this paper, the one-dimensional Ising chain plays the role as a medium to induce the effective interaction between the two external spins. Therefore, by virtue of Fröhlich transformation, we obtain the effective Hamiltonian between the two external spins by tracing over the degrees of freedom of the environment.

With an appropriate anti-Hermitian transformation S defined by the matrix elements

$$\langle m|S|n\rangle = \frac{\langle m|H_I|n\rangle}{E_n - E_m}, \quad (15)$$

which meets the condition $H_I + [H_0, S] = 0$, the effective Hamiltonian is approximated to the second order as $H_{\text{eff}} = H_C + H_{\text{el}}$, where

$$H_{\text{cl}} = \frac{1}{2} \langle 0 | [H_I, S] | 0 \rangle$$

$$= \frac{1}{2} \sum_m (\langle 0 | H_I | m \rangle \langle m | S | 0 \rangle - \langle 0 | S | m \rangle \langle m | H_I | 0 \rangle) \quad (16)$$

with $\{|m\rangle\}$ and $\{E_m\}$ being the eigenstates and eigenenergies of H_E , respectively, and $|0\rangle$ its ground state.

The DSF [18]

$$S^{\alpha\beta}(k, \omega) = \sum_{n=1}^N e^{ikn} \int_0^\infty \langle 0 | \sigma_j^\alpha(t) \sigma_{j+n}^\beta | 0 \rangle e^{(i\omega-0^+)t} dt \quad (17)$$

($\alpha, \beta = x, y$) is the Fourier transformation of the Green function for the TFIM. It is calculated as

$$S^{\alpha\beta}(k, \omega) = i \sum_{n=1}^N e^{ikn} \sum_m \frac{\langle 0 | \sigma_j^\alpha | m \rangle \langle m | \sigma_{j+n}^\beta | 0 \rangle}{E_0 - E_m + \omega + i0^+}$$

$$= i \sum_{n=1}^N e^{ikn} \sum_m \langle 0 | \sigma_j^\alpha | m \rangle \langle m | \sigma_{j+n}^\beta | 0 \rangle$$

$$\times \left[\wp \left(\frac{1}{E_0 - E_m + \omega} \right) - i\pi \delta(E_0 - E_m + \omega) \right] \quad (18)$$

with $\wp(1/x)$ being the principal value of $1/x$.

The right-hand side of Eq. (16) contains the following terms, which are expressed in terms of DSFs as

$$\sum_{m,j,j'} \langle 0 | \sigma_j^+ | m \rangle \langle m | \sigma_{j'}^+ | 0 \rangle \wp \left(\frac{1}{E_0 - E_m + \omega} \right)$$

$$= \sum_{m,j,j'} \langle 0 | \sigma_j^- | m \rangle \langle m | \sigma_{j'}^- | 0 \rangle \wp \left(\frac{1}{E_0 - E_m + \omega} \right)$$

$$= \frac{N}{4} \text{Im}[S^{xx}(0, \omega) - S^{yy}(0, \omega)], \quad (19)$$

$$\sum_{m,j,j'} \langle 0 | \sigma_j^+ | m \rangle \langle m | \sigma_{j'}^- | 0 \rangle \wp \left(\frac{1}{E_0 - E_m + \omega} \right)$$

$$= \frac{N}{4} \{ \text{Im}[S^{xx}(0, \omega) + S^{yy}(0, \omega)] - 2 \text{Re} S^{xy}(0, \omega) \}, \quad (20)$$

$$\sum_{m,j,j'} \langle 0 | \sigma_j^- | m \rangle \langle m | \sigma_{j'}^+ | 0 \rangle \wp \left(\frac{1}{E_0 - E_m + \omega} \right)$$

$$= \frac{N}{4} \{ \text{Im}[S^{xx}(0, \omega) + S^{yy}(0, \omega)] + 2 \text{Re} S^{xy}(0, \omega) \}. \quad (21)$$

Then, we rewrite the effective Hamiltonian as

$$H_{\text{eff}} = \frac{\mu_A}{2} \sigma_A^z + \frac{\mu_B}{2} \sigma_B^z + g_1 (\sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+) + g_2 (\sigma_A^+ \sigma_B^+ + \sigma_A^- \sigma_B^-) \quad (22)$$

in terms of DSFs, where [19]

$$\mu_A = \mu + J_A^2 \{ \text{Im}[S^{xx}(0, \mu) + S^{yy}(0, \mu)$$

$$- S^{xx}(0, -\mu) - S^{yy}(0, -\mu)] + 2 \text{Re}[S^{xy}(0, \mu)$$

$$+ S^{xy}(0, -\mu)] \},$$

$$\mu_B = \mu + J_B^2 \{ \text{Im}[S^{xx}(0, \mu) + S^{yy}(0, \mu) - S^{xx}(0, -\mu)$$

$$- S^{yy}(0, -\mu)] + 2 \text{Re}[S^{xy}(0, \mu) + S^{xy}(0, -\mu)] \},$$

$$g_1 = J_A J_B \{ \text{Im}[S^{xx}(0, \mu) + S^{yy}(0, \mu) + S^{xx}(0, -\mu) + S^{yy}(0, -\mu)]$$

$$+ 2 \text{Re}[S^{xy}(0, \mu) - S^{xy}(0, -\mu)] \},$$

$$g_2 = J_A J_B \text{Im}[S^{xx}(0, \mu) - S^{yy}(0, \mu) + S^{xx}(0, -\mu) - S^{yy}(0, -\mu)]. \quad (23)$$

In the forthcoming section, by using the numerical method in Ref. [18], the DSF $S^{\alpha\beta}(k, \omega)$ and thus the matrix elements of the effective Hamiltonian (23) are calculated explicitly. For further details about the diagonalization method of the Ising-like model and the fast scheme for the calculation of Pfaffian, please refer to Refs. [15,20–24].

IV. CRITICAL COUPLING

In order to calculate the DSF numerically, we shall summarize the numerical method introduced in Ref. [18]. For a spin chain in a transverse field with open ends, the Hamiltonian is described as [25]

$$H'_E = \Omega \sum_{j=1}^N \sigma_j^z + J \sum_{j=1}^{N-1} \sigma_j^x \sigma_{j+1}^x. \quad (24)$$

After the Jordan-Wigner transformation, the Hamiltonian is transformed into fermion representation. Then, it is equivalent to solving the following eigenproblem [21,23]:

$$\Phi_k (A - B)(A + B) = \Lambda_k^2 \Phi_k, \quad (25)$$

$$\Psi_k (A + B)(A - B) = \Lambda_k^2 \Psi_k, \quad (26)$$

where A and B are two $N \times N$ matrixes with their matrix elements being $A_{ij} = 2\Omega \delta_{ij} + J \delta_{i+1,j} + J \delta_{i-1,j}$ and $B_{ij} = J \delta_{i+1,j} - J \delta_{i-1,j}$. According to the Wick–Bloch–de Dominicis theorem, the x - x correlation function can be expressed in the form of the Pfaffian of the $2(2j+n-1) \times 2(2j+n-1)$ anti-symmetric matrix constructed from elementary contractions

$$\begin{aligned} \langle \sigma_j^x(t) \sigma_{j+n}^x \rangle &= \langle \varphi_1^+(t) \varphi_1^-(t) \varphi_2^+(t) \varphi_2^-(t) \cdots \varphi_{j-1}^+(t) \varphi_{j-1}^-(t) \varphi_j^+(t) \varphi_1^+ \varphi_1^- \varphi_2^+ \varphi_2^- \cdots \varphi_{j+n-1}^+ \varphi_{j+n-1}^- \varphi_{j+n}^+ \rangle \\ &= Pf \begin{bmatrix} 0 & \langle \varphi_1^+ \varphi_1^- \rangle & \langle \varphi_1^+ \varphi_2^+ \rangle & \cdots & \langle \varphi_1^+(t) \varphi_{j+n}^+ \rangle \\ -\langle \varphi_1^+ \varphi_1^- \rangle & 0 & \langle \varphi_1^- \varphi_2^+ \rangle & \cdots & \langle \varphi_1^-(t) \varphi_{j+n}^+ \rangle \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\langle \varphi_1^+(t) \varphi_{j+n}^+ \rangle & -\langle \varphi_1^-(t) \varphi_{j+n}^+ \rangle & -\langle \varphi_2^+(t) \varphi_{j+n}^+ \rangle & \cdots & 0 \end{bmatrix}, \end{aligned} \quad (27)$$

where

$$\varphi_j^\pm = c_j^\pm \pm c_j \quad (28)$$

is the linear combination of the fermion operators, and

$$\langle \varphi_j^+(t) \varphi_m^+ \rangle = \sum_{p=1}^N \Phi_{pj} \Phi_{pm} e^{-i\Lambda_p t}, \quad (29)$$

$$\langle \varphi_j^+(t) \varphi_m^- \rangle = \sum_{p=1}^N \Phi_{pj} \Psi_{pm} e^{-i\Lambda_p t}, \quad (30)$$

$$\langle \varphi_j^-(t) \varphi_m^+ \rangle = - \sum_{p=1}^N \Psi_{pj} \Phi_{pm} e^{-i\Lambda_p t}, \quad (31)$$

$$\langle \varphi_j^-(t) \varphi_m^- \rangle = - \sum_{p=1}^N \Psi_{pj} \Psi_{pm} e^{-i\Lambda_p t} \quad (32)$$

are the elementary contractions of zero temperature obtained from the finite-temperature counterparts in Refs. [18,20].

The Pfaffian is the square root of the determinant of the corresponding antisymmetric matrix. A fast computation scheme is given in Refs. [20,24]. For an $N \times N$ antisymmetric matrix

$$X = \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} \quad (33)$$

with the dimensions of A , B , C being 2×2 , $2 \times (N-2)$, $(N-2) \times (N-2)$, respectively, the Pfaffian of X can be computed in the following way. Because of

$$\begin{bmatrix} I_2 & 0 \\ B^T A^{-1} & I_{N-2} \end{bmatrix} X \begin{bmatrix} I_2 & -A^{-1} B \\ 0 & I_{N-2} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C + B^T A^{-1} B \end{bmatrix}$$

with I_n being an n -dimensional unit matrix, we have

$$\text{Det}(X) = \text{Det}(A) \text{Det}(C + B^T A^{-1} B). \quad (34)$$

Since antisymmetric A is of the simple form

$$A = \begin{bmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{bmatrix}, \quad (35)$$

the $(N-2)$ dimensional matrix $C + B^T A^{-1} B$ is also antisymmetric. The above procedure can be repeated time and time again. And the original matrix X is decomposed into $N/2$ 2D antisymmetric matrices. Finally, due to $\text{Pf}(A) = x_{12}$, the Pfaffian of matrix X will be simply a product of $N/2$ numbers

obtained from those 2×2 matrices in the above procedure. As it is not necessary for A to be a 2D matrix at the upper left corner of X , A can be chosen to be a diagonal block such that $\text{Det}(A)$ is the largest, for the stability of the algorithm.

Furthermore, other DSFs can be calculated according to the relation between correlation functions, that is,

$$\langle \sigma_j^x(t) \sigma_{j+n}^y \rangle = - \langle \sigma_j^y(t) \sigma_{j+n}^x \rangle = \frac{1}{2\Omega} \frac{d}{dt} \langle \sigma_j^x(t) \sigma_{j+n}^x \rangle, \quad (36)$$

$$\langle \sigma_j^y(t) \sigma_{j+n}^y \rangle = - \frac{1}{(2\Omega)^2} \frac{d^2}{dt^2} \langle \sigma_j^x(t) \sigma_{j+n}^x \rangle. \quad (37)$$

In the last section, we have obtained a typical spin-spin coupling in the effective Hamiltonian induced by the Ising chain. Driven by this Hamiltonian, two external spins can be entangled dynamically. To characterize the extent of entanglement, we use concurrence to measure the induced entanglement. For an arbitrary state of a two-qubit system described by the density operator ρ , a measure of entanglement can be defined as the concurrence [26,27]

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (38)$$

where the λ_i 's are the square roots of the eigenvalues of the non-Hermitian matrix $\rho \tilde{\rho}$ in decreasing order. And

$$\tilde{\rho} = (\sigma^y \otimes \sigma^y) \rho^* (\sigma^y \otimes \sigma^y), \quad (39)$$

where ρ^* is the complex conjugate of ρ .

In the following numerical simulations, we set $J=1$ as the unit of energy. As shown in Fig. 2, we investigate the evolution of concurrence under H_{eff} . The two external spins start with an initial product state $|eg\rangle$. As time passes by, the two spins will eventually evolve into a maximum entangled state. It can be seen from the figure that the time needed for reaching maximum entanglement can be greatly shortened in the vicinity of $\lambda = \Omega/J = 1$. In other words, the induced entanglement between two external spins can be enhanced by quantum criticality.

Furthermore, the relation between g_1 and λ is plotted for different N 's in Fig. 3. Although the curves oscillate sharply, they share a characteristic. There is a peak near the critical point $\lambda = 1$ for different curves. It is a reasonable result since it has been discovered that the entanglement between two nearest neighbors of the Ising chain achieves maximum near the critical point [1,2]. A similar result is also obtained for the relation between g_2 and λ in Fig. 4. However, besides the one near $\lambda = 1$, there is another peak around $\lambda = 1.5$ except that for $N=40$.

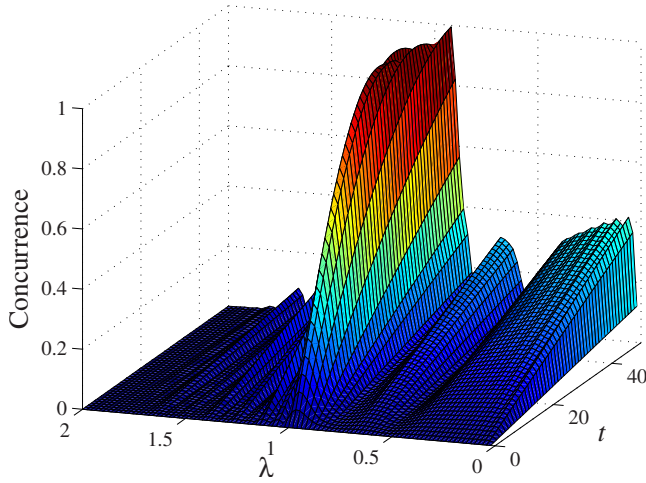


FIG. 2. (Color online) Concurrence evolution vs λ for $N=40$, $\mu=1$, $J_A=J_B=0.1$ with an initial state $|eg\rangle$.

We notice that in the above calculations, the thermodynamic limit condition is not used. As λ varies from 0 to 2, the single-particle energy spectrum goes to continuum as N increases. Since the Fröhlich transformation is equivalent to the second-order perturbation theory, it may not be valid to apply it to obtain effective interaction between the two spins. Therefore, we resort to mixed-state fidelity to demonstrate our approximation used in this paper. The mixed-state fidelity is given as [28,29]

$$F(\rho_0, \rho_1) = \text{tr} \sqrt{\rho_1^{1/2} \rho_0 \rho_1^{1/2}}, \quad (40)$$

which measures the degree of distinguishability between the two quantum states ρ_0 and ρ_1 . It has already been applied to the research on QPT [30]. Starting from the original Hamiltonian (1), we obtain the reduced density matrix ρ_0 for the ground state of the two spins by first diagonalizing H and then tracing over all the degrees of the Ising chain. On the other hand, we can also obtain the density matrix ρ_1 of the

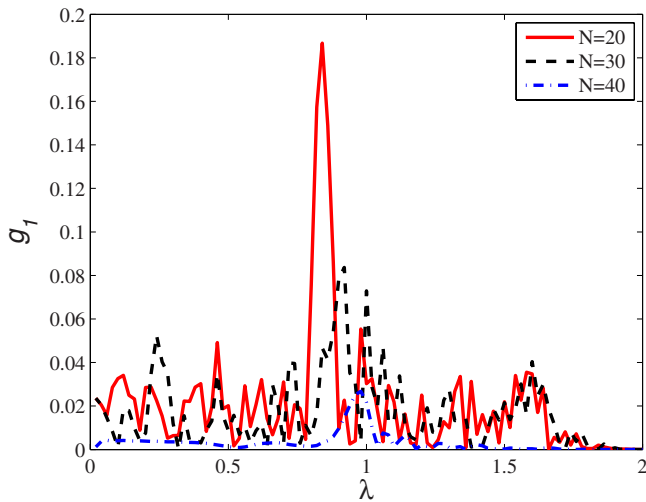


FIG. 3. (Color online) g_1 (in unit of J) vs λ for different N 's with $\mu=1$ and $J_A=J_B=0.1$. Red solid line for $N=20$, black dashed line for $N=30$, blue dash-dot line for $N=40$.

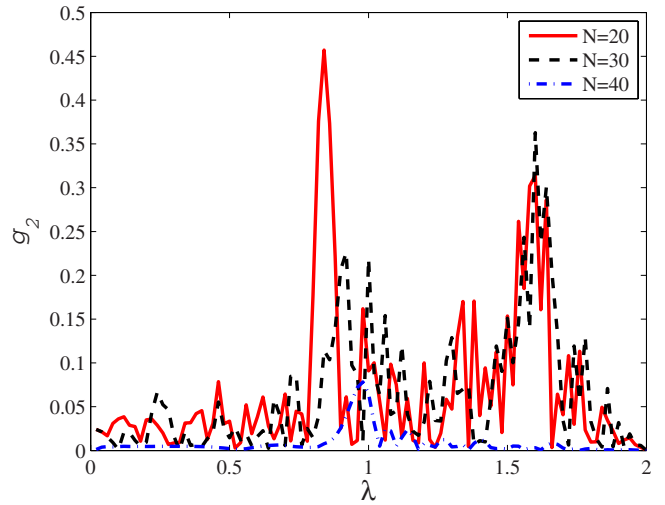


FIG. 4. (Color online) g_2 (in unit of J) vs λ for different N 's with $\mu=1$ and $J_A=J_B=0.1$. Red solid line for $N=20$, black dashed line for $N=30$, blue dash-dot line for $N=40$.

ground state from the effective Hamiltonian (22) of the two spins. In Fig. 5, the relation between fidelity and λ is plotted for different N 's. Despite oscillations in some parameter intervals, the numerical method shows high fidelity over the whole interval. Moreover, curves of different N 's converge as λ goes larger.

Here, we state that the oscillations in Figs. 3–5 are due to the removal of resonance points in obtaining the effective Hamiltonian (22). And this approximation leads to oscillations in the figure of fidelity vs λ (see Fig. 5), which shows the validity of our approximation and numerical simulations.

V. CONCLUSION

In summary, we have studied the dynamical process of two external spins simultaneously coupling to a transverse

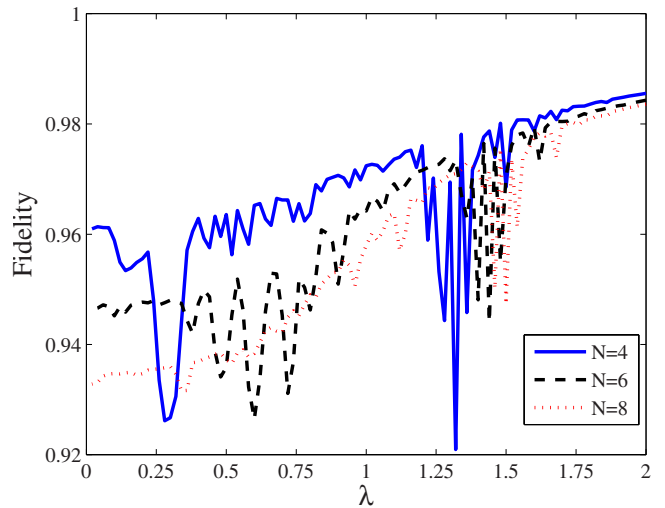


FIG. 5. (Color online) Fidelity vs λ for different N 's with $\mu=1$ and $J_A=J_B=0.1$. Blue solid line for $N=4$, black dashed line for $N=6$, red dotted line for $N=8$.

field Ising chain. With Fröhlich transformation, we have deduced the effective Hamiltonian between these two spins. The matrix elements of the Hamiltonian are expressed in terms of the DSF, which can be numerically calculated. Through the numerical simulation, it is shown that the induced entanglement is enhanced by QPT. And the effective coupling constants reach maximum near the critical point. By virtue of mixed-state fidelity, we demonstrate the validity of the Fröhlich transformation and numerical simulation. Thus, the measure of the two spin entanglement can be an illustration of QPT.

Besides the enhancement of the coupling intensity around the critical point, there are oscillations elsewhere. We remark that QPT takes place in the thermodynamic limit, i.e., $N \rightarrow \infty$. In this limit, the energy spectrum of TFIM goes to continuum from zero to infinity at some parameter, i.e., $\lambda = 1$. Therefore, the eigenenergy of the external spins will definitely be resonant with one of the eigenenergies of the Ising chain. In this circumstance, our method may not work well. We notice that a numerical method, the time-dependent

density matrix renormalization group (*t*-DMRG) [31], was applied to central spin models, which were quite similar to ours. Therefore, in the near future, we will apply this method to our model to obtain a better result. We expect smooth curves similar to Fig. 1 in Ref. [1].

Although our analytical derivation and numerical simulation are based on the TFIM, these results are applicable to other spin chain models of QPT, i.e., the *XY* model, since the TFIM is a special case in which $\gamma=1$, which describes the anisotropy of the interaction between the environmental spins.

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